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2013 Izv. Math. 77 378

(http://iopscience.iop.org/1064-5632/77/2/378)

Izvestiya: *Mathematics* **77**:2 378–406 © 2013 RAS(DoM) and LMS
Izvestiya RAN: *Ser. Mat.* **77**:2 165–196 DOI 10.1070/IM2013v077n02ABEH002640

Quantum field theories on algebraic curves. I. Additive bosons

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Abstract. Using Serre's adelic interpretation of cohomology, we develop a 'differential and integral calculus' on an algebraic curve X over an algebraically closed field k of constants of characteristic zero, define algebraic analogues of additive multi-valued functions on X and prove the corresponding generalized residue theorem. Using the representation theory of the global Heisenberg algebra and lattice Lie algebra, we formulate quantum field theories of additive and charged bosons on an algebraic curve X. These theories are naturally connected with the algebraic de Rham theorem. We prove that an extension of global symmetries (Witten's additive Ward identities) from the k-vector space of rational functions on X to the vector space of additive multi-valued functions uniquely determines these quantum theories of additive and charged bosons.

Keywords: algebraic curves and algebraic functions, adèles, additive multi-valued functions, additive Ward identities, Heisenberg algebra, current algebra on an algebraic curve, generalized residue theorem, Fock spaces, quantum theories of free bosons on an algebraic curve, expectation value functional.

§ 1. Introduction

The classical theory of compact Riemann surfaces has an algebraic counterpart, the theory of algebraic functions of one variable over an arbitrary field of constants, as developed by Dedekind and Weber. The introduction of differentials in the algebraic theory by Artin and Hasse and the definition of idèles and adèles by Chevalley and Weil opened the way for the application of infinite-dimensional methods to the theory of algebraic curves. Classical examples of using such methods are given by Serre's adelic interpretation of cohomology and the Riemann–Roch theorem [1] and Tate's proof of the general residue theorem [2]. In 1987, Arbarello, de Concini and Kac [3] interpreted Tate's approach in terms of central extensions of infinite-dimensional Lie algebras and gave a new proof of Weil's celebrated reciprocity law using the infinite wedge representation.

In 1987, Kazhdan [4] and Witten [5] proposed an adelic formulation of the quantum field theory of one-component free fermions on an algebraic curve, and Witten [6] outlined an approach to other quantum field theories. Let X be an

This paper was written with the financial support of NSF (grants nos. DMS-0204628, DMS-0705263 and DMS-1005769).

AMS 2010 *Mathematics Subject Classification.* 81R10, 14H81.

algebraic curve over an algebraically closed field k of constants, and let L be a spin structure on X. We write $\mathcal{M}(L)$ for the infinite-dimensional k-vector space of meromorphic sections of L over X, and $\mathcal{M}_P$ for the completions of $\mathcal{M}(L)$ at all points $P \in X$. In outline, the approach of [4], [5] can be described in terms of the following objects.

1) The global Clifford algebra Cl_X on X: a restricted direct product over all points $P \in X$ of the local Clifford algebras Cl_P, which are related to the k-vector spaces $\mathcal{M}_P$ by the residue maps $\mathrm{Res}_P(fg)$.

2) The adelic Clifford module (the global fermion Fock space $\mathfrak{F}_X$): a restricted $\mathbb{Z}/2\mathbb{Z}$-graded tensor product of the local Clifford modules $\mathfrak{F}_P$ over all $P \in X$.

3) The 'expectation value' functional: a linear map $\langle\cdot\rangle\colon \mathfrak{F}_X \to k$ satisfying the condition

$$\langle f \cdot u\rangle = 0 \quad \forall\, f \in \mathcal{M}(L) \subset \mathrm{Cl}_X, \qquad u \in \mathfrak{F}_X, \tag{1}$$

where the vector space $\mathcal{M}(L)$ is embedded diagonally in the global Clifford algebra Cl_X.

In this purely algebraic formulation of one-component free fermions on an algebraic curve, the products of the field operators at points $P \in X$ are replaced by the vectors $u = \widehat{\otimes}_{P\in X} u_P \in \mathfrak{F}_X$, and the linear map $\langle\cdot\rangle$ is a mathematical way of defining the correlation functions of quantum fields. At the physical level of rigour, these functions are introduced by the Feynman path integral. The vector space $\mathcal{M}(L)$ acts on $\mathfrak{F}_X$ by global symmetries, and the invariance of the quantum theory of free fermions with respect to these symmetries is expressed by the quantum conservation laws (1), also known as the *additive Ward identities*. It is proved in [5], [6] that if the spin structure L has no global holomorphic sections, then the additive Ward identities uniquely determine the expectation value functional $\langle\cdot\rangle$. The relations (1) are compatible with the global residue theorem on X:

$$\sum_{P\in X} \mathrm{Res}_P(f\,dg) = 0, \qquad f, g \in \mathcal{M}(L).$$

Witten [6] developed the basics of quantum field theories associated with current algebras on algebraic curves and mentioned the theories associated with loop groups on algebraic curves. The global symmetries of these theories are respectively given by the rational maps of the algebraic curve X to a finite-dimensional semisimple Lie algebra over k and the rational maps of X to the corresponding Lie group. In the latter case, the analogues of quantum conservation laws (1) were called *multiplicative Ward identities* in [6]. It was emphasized in [6], § IV that if the genus of X is greater than zero, then the Ward identities do not uniquely determine the expectation value functional $\langle\cdot\rangle$, even in the Lie-algebraic case. Thus the main problem in the construction of quantum field theories on an algebraic curve is to find additional conditions which would uniquely determine the linear functional $\langle\cdot\rangle$.

When X is a Riemann surface (that is, an algebraic curve over the field $\mathbb{C}$ of complex numbers, equipped with the complex topology), the usual physicist's representation of correlation functions is given by the Feynman path integral, which uses the Lagrangian formulation of the theory. This approach is not applicable in the case when X is an algebraic curve over an arbitrary field of constants. Hence

one needs other ways to define the correlation functions. The basic example is given by the 'integrable' case, when the expectation value functional is uniquely determined by the global symmetries (and hence so are all correlation functions).

In [7] we gave a solution of the problem of the unique determination of the expectation value functional for the simplest scalar theories, when the finite-dimensional Lie algebra is the Abelian Lie algebra k, and the corresponding Lie group is the multiplicative group $k^* = k \setminus \{0\}$. We call these quantum field theories the theories of *additive* and *multiplicative bosons* respectively. The solution suggested in [7] involves enlarging the global symmetries by considering the algebraic analogue of the vector space of additive multi-valued functions on a Riemann surface (analogues of the classical Abelian integrals of the second kind with zero a-periods). Although the classical theory of Abelian integrals was already developed by Riemann (see, for example, [8] and [9] for a modern exposition), the corresponding algebraic theory (integral calculus on algebraic curves) has not been fully developed. In this paper we partially fill this gap in the case when the field k of constants has characteristic zero and give an explicit construction of the quantum field theories of additive bosons on an algebraic curve. These theories are naturally connected with the algebraic de Rham theorem, and their global symmetries form a vector space of additive multi-valued functions; see Theorems 6, 7 for precise statements. Our construction of quantum field theories on algebraic curves may be regarded as an algebraic analogue of the geometric realization of conformal field theories on Riemann surfaces [10]. The quantum field theory of multiplicative bosons requires an algebraic analogue of the group of multiplicative multi-valued functions on a Riemann surface (analogues of the exponentials of Abelian integrals of the third kind with zero a-periods). We plan to discuss the analogue of this group and the corresponding multiplicative Ward identities in a separate publication.

Here is a more detailed description of the contents of the paper. In § 2 we recall the necessary basic facts from the theory of algebraic curves. Namely, let X be an algebraic curve of genus g over an algebraically closed field k of constants, $F = k(X)$ the field of rational functions on X, and F_P the corresponding local fields (the completions of F with respect to the regular valuations v_P corresponding to the discrete valuation rings at the points $P \in X$). In § 2.1 we introduce the ring of adèles

$$\mathbb{A}_X = \coprod_{P \in X} F_P$$

as a restricted direct product of the local fields F_P and describe Serre's adelic interpretation of cohomology. In § 2.2 we recall the definitions of the F-module $\Omega^1_{F/k}$ of Kähler differentials on X, the corresponding $\mathbb{A}_X$-module $\boldsymbol{\Omega}_X$ of differential adèles, the differential map $d\colon \mathbb{A}_X \to \boldsymbol{\Omega}_X$ and the residue map $\operatorname{Res}\colon \boldsymbol{\Omega}_X \to k$. In § 2.3 we describe Serre duality and the Riemann–Roch theorem.

In § 3, assuming that the field k of constants is of characteristic zero, we recall the differential calculus on an algebraic curve X (the structural theory of the k-vector space $\Omega^1_{F/k}$ of Kähler differentials on X) and develop a corresponding integral calculus. Namely, in § 3.1 we follow [11] and [12] and endow the k-vector space $\Omega^{(2\mathrm{nd})}$ of differentials of the second kind (that is, differentials on X with zero residues)

with a skew-symmetric bilinear form

$$(\omega_1, \omega_2)_X = \sum_{P \in X} \mathrm{Res}_P(d^{-1}\omega_1\,\omega_2), \qquad \omega_1, \omega_2 \in \Omega^{(2\mathrm{nd})}.$$

The main result of the differential calculus is Theorem 4, an algebraic version of de Rham's theorem. Theorem 4 goes back to Chevalley and Eichler and, for an algebraic curve X of genus $g \geqslant 1$, states [1] that the $2g$-dimensional k-vector space $\Omega^{(2\mathrm{nd})}/dF$ is a symplectic vector space with symplectic form $(\ ,\ )_X$. Moreover, for every choice of a non-special effective divisor $D = P_1 + \cdots + P_g$ of degree g on X and uniformizers t_i at P_i, there is an isomorphism

$$\Omega^{(2\mathrm{nd})}/dF \simeq \Omega^{(2\mathrm{nd})} \cap \Omega^1_{F/k}(2D).$$

The space $\Omega^{(2\mathrm{nd})} \cap \Omega^1_{F/k}(2D)$ has a natural symplectic basis $\{\theta_i, \omega_i\}_{i=1}^g$, where θ_i (resp. ω_i) are differentials of the first (resp. second) kind with the following properties. The θ_i vanish at all points P_j, $j \neq i$, and $\theta_i = (1 + O(t_i))dt_i$ at P_i, while the ω_i are regular at all points P_j, $j \neq i$, and $\omega_i = (t_i^{-2} + O(t_i))dt_i$ at P_i. Hence the θ_i (resp. ω_i) are algebraic analogues of the differentials of the first kind with normalized a-periods (resp. differentials of the second kind with second-order poles, zero a-periods and normalized b-periods) on a compact Riemann surface. The a-periods of $\omega \in \Omega^{(2\mathrm{nd})}$ are algebraically defined as $(\omega, \omega_i)_X$, $i = 1, \ldots, g$, and we write $\Omega_0^{(2\mathrm{nd})}$ for the isotropic subspace of $\Omega^{(2\mathrm{nd})}$ consisting of all differentials of the second kind with zero a-periods. By Proposition 1 we have

$$\Omega_0^{(2\mathrm{nd})} = k \cdot \omega_1 \oplus \cdots \oplus k \cdot \omega_g \oplus dF. \tag{2}$$

In § 3.1 we also introduce an algebraic notion of additive multi-valued functions on X. By definition, the k-vector space of additive multi-valued functions is a subspace $\mathcal{A}(X)$ of the adèle ring $\mathbb{A}_X$ satisfying $F \subset \mathcal{A}(X)$ and $d\mathcal{A}(X) \subset \Omega^1_{F/k}$ and the additional condition that if $a \in \mathcal{A}(X)$ and $da = 0$, then $a = c \in k$. The main result of the integral calculus for differentials of the second kind with zero a-periods is the explicit construction of the vector space $\mathcal{A}(X, D)$ in Example 1. This space plays a fundamental role in the theory of additive bosons. It is parametrized by the choices of a non-special divisor $D = P_1 + \cdots + P_g$ of degree g on X, local uniformizers t_i at the points P_i and solutions of the equations $d\eta_i = \omega_i$ in $\mathbb{A}_X$ (with any fixed choice of the local additive constants). It is defined as

$$\mathcal{A}(X; D) = k \cdot \eta_1 \oplus \cdots \oplus k \cdot \eta_g \oplus F \subset \mathbb{A}_X,$$

and possesses the property $d(\mathcal{A}(X; D)) = \Omega_0^{(2\mathrm{nd})}$. We finally introduce the additive multi-valued functions $\eta_P^{(n)} \in \mathcal{A}(X; D)$ with a single pole of order n at $P \in X$ and prove (see Lemma 1) that every rational function $f \in F$ has a unique partial fraction expansion, the partial fractions being these $\eta_P^{(n)}$. We also explain the difficulties arising in an attempt to define algebraic analogues of multiplicative multi-valued functions.

[1] The case $g = 0$ is trivial.

In § 4 we formulate the local quantum field theories of additive and charged bosons. The local theory of additive bosons is associated with the representation theory of the local Heisenberg algebra $\mathfrak{g}_P$, a one-dimensional central extension of the Abelian Lie algebra F_P, $P \in X$, by the 2-cocycle $c_P(f,g) = -\operatorname{Res}_P(f dg)$. In § 4.1 we introduce the highest-weight representation ρ of $\mathfrak{g}_P$ on the local Fock space $\mathscr{F}_P$ and define the corresponding contragradient representation $\rho^\vee$ of $\mathfrak{g}_P$ on the dual local Fock space $\mathscr{F}_P^\vee$. In § 4.2 we define a local lattice algebra $\mathfrak{l}_P$ as a semi-direct sum of the local Heisenberg algebra $\mathfrak{g}_P$ and the Abelian Lie algebra $k[\mathbb{Z}]$, the group algebra of $\mathbb{Z}$. The corresponding irreducible highest-weight $\mathfrak{l}_P$-module is the local Fock space $\mathscr{B}_P$ of 'charged bosons' (the tensor product of $k[\mathbb{Z}]$ and $\mathscr{F}_P$). The material in §§ 4.1, 4.2 is essentially standard (see [13], [14]).

In § 5 we finally state the global quantum field theories, starting in § 5.1 with the theory of additive bosons on an algebraic curve X. This theory is naturally connected with the global Heisenberg algebra $\mathfrak{g}_X$, a one-dimensional central extension of the Abelian Lie algebra $\mathfrak{gl}_1(\mathbb{A}_X) = \mathbb{A}_X$ by the 2-cocycle $c_X = \sum_{P\in X} c_P$. Since the subspace $\Omega_0^{(2\mathrm{nd})}$ is isotropic with respect to the bilinear form $(\ ,\ )_X$, we have

$$c_X(a_1, a_2) = 0 \qquad \forall\, a_1, a_2 \in \mathcal{A}(X, D) \subset \mathbb{A}_X. \tag{3}$$

This may be regarded as a generalized residue theorem for additive multi-valued functions. The irreducible highest-weight module of the global Heisenberg algebra $\mathfrak{g}_X$ is the global Fock space $\mathscr{F}_X$, a restricted tensor product of the local Fock spaces $\mathscr{F}_P$ over all points $P \in X$. It may be regarded as the space of observables of the quantum theory of additive bosons on X. In Theorem 6 we prove that there is a unique normalized expectation value functional $\langle\,\cdot\,\rangle\colon \mathscr{F}_X \to k$, uniquely characterized by the global symmetries

$$\langle \rho(a) v\rangle = 0 \quad \forall\, a \in \mathcal{A}(X; D), \qquad v \in \mathscr{F}_X. \tag{4}$$

Here $\mathcal{A}(X;D) \subset \mathbb{A}_X$ is the vector space of additive multi-valued functions on X defined in § 3.1, and $\rho\colon \mathfrak{g}_X \to \operatorname{End}\mathscr{F}_X$ is the corresponding representation of the global Heisenberg algebra. Specifically, we show in Theorem 6 that

$$\langle v\rangle = (\Omega_X, v) \qquad \forall\, v \in \mathscr{F}_X,$$

where $\Omega_X \in \mathscr{F}_X^\vee$ is a vector in the space dual to $\mathscr{F}_X$ satisfying an infinite system of equations

$$\Omega_X \cdot \rho^\vee(a) = 0 \qquad \forall\, a \in \mathcal{A}(X, D). \tag{5}$$

The vector Ω_X is given by an explicit formula (Theorem 6), which encodes the analogues of all correlation functions of quantum additive bosons on X. The compatibility of the system (5) is based on the reciprocity law (proved in Lemma 1) for the differentials of the second kind with zero a-periods.

The additive Ward identities (4) are also compatible with the generalized residue theorem. Namely, since $[\rho(x), \rho(y)] = c_X(x,y)\mathbf{I}$ for $x, y \in \mathbb{A}_X$, where $\mathbf{I}$ is the identity operator on $\mathscr{F}_X$, we get from (4) that for $a_1, a_2 \in \mathcal{A}(X, D)$,

$$0 = \big\langle (\rho(a_1)\rho(a_2) - \rho(a_2)\rho(a_1))v\big\rangle = c_X(a_1, a_2)\langle v\rangle \qquad \forall\, v \in \mathscr{F}_X,$$

which yields (3).

In § 5.2 we define a global lattice algebra $\mathfrak{l}_X$ as a semi-direct sum of the global Heisenberg algebra $\mathfrak{g}_X$ and the Abelian Lie algebra $k[\mathrm{Div}_0(X)]$ with generators e_D, $D \in \mathrm{Div}_0(X)$, where $k[\mathrm{Div}_0(X)]$ is the group algebra of the additive group $\mathrm{Div}_0(X)$ of divisors of degree 0 on X. Its irreducible highest-weight module is the global Fock space $\mathscr{B}_X$ of charged bosons, which is the tensor product of the group algebra $k[\mathrm{Div}_0(X)]$ and the Fock space $\mathscr{F}_X$ of additive bosons. The main result of § 5.2 is Theorem 7 on the existence and uniqueness of an expectation value functional $\langle\cdot\rangle\colon \mathscr{B}_X \to k$ which is normalized with respect to the action of the group algebra $k[\mathrm{Div}_0(X)]$ and satisfies the additive Ward identities (4) with respect to the action of the global symmetries (additive multi-valued functions in $\mathcal{A}(X, D)$) on the global Fock space $\mathscr{B}_X$. This functional is of the form $\langle v\rangle = (\widehat{\Omega}_X, v)$, where the vector $\widehat{\Omega}_X \in \mathscr{B}_X^\vee$ in the space dual to $\mathscr{B}_X$ is given by an explicit formula (see Theorem 7), which encodes all correlation functions of quantum charged additive bosons on X. In § 5.3, following a suggestion of the referee, we give a more invariant formulation of Theorem 6.

I am grateful to the referee for his careful reading of the manuscript, constructive criticism, remarks, and valuable suggestions.

§ 2. Basic facts

Here we recall necessary facts from the theory of algebraic curves. This material is essentially standard (see [1], [8], [11]).

2.1. Definitions. An algebraic curve X over an algebraically closed field k is an irreducible non-singular one-dimensional projective variety over k. It is equipped with the Zariski topology. The field $F = k(X)$ of rational functions on X is a finitely generated extension of k of transcendence degree 1. Conversely, every finitely generated extension of k of transcendence degree 1 corresponds to a unique (up to isomorphism) algebraic curve over k. Closed points P on X correspond to discrete valuation rings O_P (subrings of F). The rings O_P for all $P \in X$ form a sheaf of rings on X: the structure sheaf $\mathcal{O}_X$, a subsheaf of the constant sheaf $\underline{F}$.

For every point $P \in X$ let v_P be the regular discrete valuation of F over k corresponding to the discrete valuation ring O_P. The completion of F with respect to v_P is a complete closed field F_P with valuation ring $\mathcal{O}_P$, which is the completed local ring at P with prime ideal $\mathfrak{p}$ and residue class field $k = \mathcal{O}_P/\mathfrak{p}$. The ring $\mathbb{A}_X$ of adèles of X is

$$\mathbb{A}_X = \coprod_{P\in X} F_P,$$

the restricted direct product over all points $P \in X$ of the local fields F_P with respect to the local rings $\mathcal{O}_P$. By definition,

$$x = \{x_P\}_{P\in X} \in \mathbb{A}_X$$

if $x_P \in \mathcal{O}_P$ for all but finitely many $P \in X$. The field F is contained in all local fields F_P and is diagonally embedded in $\mathbb{A}_X$:

$$F \ni f \mapsto \{f|_P\}_{P\in X} \in \mathbb{A}_X.$$

The divisor group $\mathrm{Div}(X)$ of X is the free Abelian group generated by the points $P \in X$. By definition,

$$D = \sum_{P \in X} n_P \cdot P \in \mathrm{Div}(X)$$

if $n_P = v_P(D) \in \mathbb{Z}$ and $n_P = 0$ for all but finitely many $P \in X$. Divisors of the form

$$(f) = \sum_{P \in X} v_P(f) \cdot P \in \mathrm{Div}(X),$$

where $f \in F^* = F \setminus \{0\}$, are called *principal divisors*. They form a subgroup $\mathrm{PDiv}(X) \simeq F^*/k^*$ of $\mathrm{Div}(X)$. The degree of a divisor D is

$$\deg D = \sum_{P \in X} n_P = \sum_{P \in X} v_P(D) \in \mathbb{Z},$$

and we have $\deg(f) = 0$ if $f \in F^*$. A divisor D is said to be *effective* if $v_P(D) \geqslant 0$ for all $P \in X$. By definition, D_1 and D_2 are linearly equivalent ($D_1 \sim D_2$) if $D_1 - D_2 = (f)$ for some $f \in F^*$. The equivalence classes of divisors form the divisor class group $\mathrm{Cl}(X) = \mathrm{Div}(X)/\mathrm{PDiv}(X)$.

For every divisor D we define a subspace $\mathbb{A}_X(D)$ of the k-vector space $\mathbb{A}_X$ by putting

$$\mathbb{A}_X(D) = \{x \in \mathbb{A}_X : v_P(x_P) \geqslant -v_P(D) \ \forall\, P \in X\}.$$

The ring $\mathbb{A}_X$ of adèles is a topological ring with the product topology. A base of neighbourhoods of 0 is given by the subspaces $\mathbb{A}_X(D)$, $D \in \mathrm{Div}(X)$, and $\mathbb{A}_X$ is a k-vector space with linear topology in the sense of Lefschetz ([15], Ch. II, § 6). Since the subspaces $\mathbb{A}_X(D)$ are linearly compact, $\mathbb{A}_X$ is locally linearly compact. The k-vector space $F = k(X)$ is discrete in $\mathbb{A}_X$ and the quotient space $\mathbb{A}_X/F$ is linearly compact ([8], Appendix, § 3).

For every divisor D we have an algebraic coherent sheaf $\mathcal{O}_X(D)$ on X whose stalk at any point $P \in X$ is

$$\mathcal{O}_X(D)_P = \{f \in F : v_P(f) \geqslant -v_P(D)\}.$$

Linearly equivalent divisors correspond to isomorphic sheaves. We denote the Čech cohomology groups of the sheaf $\mathcal{O}_X(D)$ by $H^i(X, \mathcal{O}_X(D))$ (these are finite-dimensional vector spaces over k, trivial for $i > 1$) and put $h^i(D) = \dim_k H^i(X, \mathcal{O}_X(D))$. The zero divisor $D = 0$ corresponds to the structure sheaf $\mathcal{O}_X$. In this case, $h^0(0) = 1$ and $h^1(0) = g$ is the arithmetic genus of X. We have

$$H^0(X, \mathcal{O}_X(D)) = \mathbb{A}_X(D) \cap F, \qquad H^1(X, \mathcal{O}_X(D)) \simeq \mathbb{A}_X/(\mathbb{A}_X(D) + F),$$

which is Serre's adelic interpretation of cohomology ([1], Ch. II, § 5).

2.2. Differentials and residues. Let R be a ring over k. The module $\Omega^1_{R/k}$ of Kähler differentials of R is the universal R-module with the property that there is a k-linear map $d : R \to \Omega^1_{R/k}$ satisfying the Leibniz rule

$$d(fg) = f\,dg + g\,df, \qquad f, g \in R.$$

When $R = F$ is the field $k(X)$ of rational functions on an algebraic curve X, $\Omega^1_{F/k}$ is a one-dimensional vector space over F. Let $t \in F$ be a local coordinate at P in the Zariski topology, that is, a rational function on X with $v_P(t) = 1$. Then dt is a basis of the F-vector space $\Omega^1_{F/k}$, that is, every Kähler differential can be written as $\omega = f dt$ for some $f \in F$. The order of $\omega \in \Omega^1_{F/k}$ at P is defined by

$$v_P(\omega) = v_P(f).$$

It is independent of the choice of the local coordinate at P and determines a valuation on $\Omega^1_{F/k}$.

The family of O_P-modules $\Omega^1_{O_P/k}$ for all points $P \in X$ forms an algebraic coherent sheaf $\underline{\Omega}$, a subsheaf of the constant sheaf $\underline{\Omega^1_{F/k}}$. Moreover,

$$\Omega^1_{F/k} = \Omega^1_{O_P/k} \otimes_{O_P} F.$$

When k has characteristic 0, the F_P-module $\Omega^1_{F_P/k}$ is an infinite-dimensional vector space over F_P for every point $P \in X$ (the map d is not continuous with respect to the $\mathfrak{p}$-adic topology on F_P). Following [1], Ch. II, § 11, we define

$$\widetilde{\Omega}^1_{F_P/k} = \Omega^1_{F_P/k}/\mathcal{Q},$$

where $\mathcal{Q} = \bigcap_{n \geqslant 0} \mathfrak{p}^n d(\mathcal{O}_P)$ and, therefore, $\dim_{F_P} \widetilde{\Omega}^1_{F_P/k} = 1$. The resulting F_P-module $\widetilde{\Omega}^1_{F_P/k}$ is the completion of the F-module $\Omega^1_{F/k}$ with respect to the valuation v_P. The completion of the O_P-module $\Omega^1_{O_P/k}$ is the $\mathcal{O}_P$-module $\widetilde{\Omega}^1_{\mathcal{O}_P/k}$ and

$$\widetilde{\Omega}^1_{F_P/k} = \widetilde{\Omega}^1_{\mathcal{O}_P/k} \otimes_{\mathcal{O}_P} F_P.$$

We define the $\mathbb{A}_X$-module $\boldsymbol{\Omega}_X$ of differential adèles of the sheaf $\underline{\Omega}$ by the formula

$$\boldsymbol{\Omega}_X = \coprod_{P \in X} \widetilde{\Omega}^1_{F_P/k}.$$

This is the restricted direct product over all points $P \in X$ of the F_P-modules $\widetilde{\Omega}^1_{F_P/k}$ with respect to the $\mathcal{O}_P$-modules $\widetilde{\Omega}^1_{\mathcal{O}_P/k}$. The F-module $\Omega^1_{F/k}$ is contained in all F_P-modules $\widetilde{\Omega}^1_{F_P/k}$ and is diagonally embedded in $\boldsymbol{\Omega}_X$:

$$\Omega^1_{F/k} \ni \omega \mapsto \{\omega|_P\}_{P \in X} \in \boldsymbol{\Omega}_X.$$

The k-vector space $\boldsymbol{\Omega}_X$ has a linear topology with a base of neighbourhoods of zero given by the subspaces $\boldsymbol{\Omega}_X(D)$ for all $D \in \mathrm{Div}(X)$:

$$\boldsymbol{\Omega}_X(D) = \big\{\omega = \{\omega_P\}_{P \in X} \in \boldsymbol{\Omega}_X \colon v_P(\omega_P) \geqslant -v_P(D)\ \forall\, P \in X\big\}.$$

This topological space is locally linearly compact. The maps $d\colon F_P \to \widetilde{\Omega}^1_{F_P/k}$ for all $P \in X$ determine a continuous map $d\colon \mathbb{A}_X \to \boldsymbol{\Omega}_X$ satisfying the Leibniz rule.

Remark 1. The $\mathbb{A}_X$-module $\boldsymbol{\Omega}_X$ is essentially the set of 'principal part systems of degree 1' on X in the sense of Eichler (see [12], Ch. III, § 5.2).

Take $\omega \in \widetilde{\Omega}^1_{F_P/k}$ and let t be a local parameter of the field F_P, so that dt is a basis of the $\mathcal{O}_P$-module $\widetilde{\Omega}^1_{\mathcal{O}_P/k}$. The residue map $\mathrm{Res}_P\colon \widetilde{\Omega}^1_{F_P/k} \to k$ is defined as

$$\mathrm{Res}_P(\omega) = c_{-1}, \quad \text{where} \quad \omega = \sum_{n \gg -\infty}^{+\infty} c_n t^n\, dt,$$

and the symbol $n \gg -\infty$ means that the summation is taken over only finitely many negative values of n. The definition of the residue is independent of the choice of the local parameter. The residue map is continuous with respect to the $\mathfrak{p}$-adic topology on $\widetilde{\Omega}^1_{F_P/k}$ and the discrete topology on k. The local residue maps Res_P give rise to the global residue map $\mathrm{Res}\colon \boldsymbol{\Omega}_X \to k$,

$$\mathrm{Res}\,\omega = \sum_{P \in X} \mathrm{Res}_P(\omega_P), \qquad \omega = \{\omega_P\}_{P \in X} \in \boldsymbol{\Omega}_X.$$

The global residue map is well defined, continuous and possesses the following fundamental property.

Theorem 1 (the residue formula). *For every* $\omega \in \Omega^1_{F/k}$,

$$\mathrm{Res}\,\omega = \sum_{P \in X} \mathrm{Res}_P(\omega|_P) = 0.$$

2.3. Serre duality and the Riemann–Roch theorem. We put

$$\Omega^1_{F/k}(D) = \Omega^1_{F/k} \cap \boldsymbol{\Omega}_X(D) = \{\omega \in \Omega^1_{F/k}\colon v_P(\omega) \geqslant -v_P(D)\ \forall\, P \in X\}$$

and define the residue pairing $(\ ,\ )\colon \boldsymbol{\Omega}_X \bigotimes_K \mathbb{A}_X \to k$ by the formula

$$(\omega, x) = \sum_{P \in X} \mathrm{Res}_P(x_P \omega_P), \quad \text{where} \quad \omega \in \boldsymbol{\Omega}_X, \quad x \in \mathbb{A}_X.$$

The residue pairing has the following properties:

P1) $(\omega, x) = 0$ if $\omega \in \Omega^1_{F/k}$ and $x \in F$,

P2) $(\omega, x) = 0$ if $\omega \in \boldsymbol{\Omega}_X(-D)$ and $x \in \mathbb{A}_X(D)$.

It follows from P1), P2) that for every $D \in \mathrm{Div}(X)$ the formula $\imath(\omega)(x) = (\omega, x)$ determines a k-linear map

$$\imath\colon \Omega^1_{F/k}(-D) \to \big(\mathbb{A}_X/(\mathbb{A}_X(D) + F)\big)^\vee,$$

where $V^\vee = \mathrm{Hom}(V, k)$ is the topological dual of a k-vector space V with linear topology.

Theorem 2 (Serre duality). *For every* $D \in \mathrm{Div}(X)$, *the map* $\imath$ *is an isomorphism. Hence the finite-dimensional* k*-vector spaces* $\Omega^1_{F/k}(-D)$ *and* $\mathbb{A}_X/(\mathbb{A}_X(D) + F)$ *are dual with respect to the residue pairing.*

Corollary 1 (the strong residue theorem).

(i) *An adèle* $x \in \mathbb{A}_X$ *corresponds to a rational function on* X *under the embedding* $F \hookrightarrow \mathbb{A}_X$ *if and only if* $(\omega, x) = 0$ *for all* $\omega \in \Omega^1_{F/k}$.

(ii) *A differential adèle* $\omega \in \boldsymbol{\Omega}_X$ *corresponds to a Kähler differential on* X *under the embedding* $\Omega^1_{F/k} \hookrightarrow \boldsymbol{\Omega}_X$ *if and only if* $(\omega, f) = 0$ *for all* $f \in F$.

Proof. Suppose that the condition in (i) holds. It follows from Serre duality that $x \in \mathbb{A}_X(D) + F$ for every $D \in \operatorname{Div}(X)$ and, since $F \cap \mathbb{A}_X(D) = 0$ for $D < 0$, we have $x \in F$. To prove (ii), take $\omega_0 \in \Omega^1_{F/k}$, $\omega_0 \neq 0$. Putting $x = \omega/\omega_0 \in \mathbb{A}_X$, we have $0 = (\omega, f) = (f\omega_0, x)$ for all $f \in F$, whence $x \in F$ by part (i). □

Remark 2. The strong residue theorem is stated in a slightly different form in [12], Ch. III, § 5.3.

For $\omega \in \Omega^1_{F/k}$ we put

$$(\omega) = \sum_{P \in X} v_P(\omega) \cdot P \in \operatorname{Div}(X).$$

Since $\dim_F \Omega^1_{F/k} = 1$, all divisors (ω) are linearly equivalent and determine a divisor class $K \in \operatorname{Cl}(X)$, the canonical class of X. The following result is obtained by combining the Riemann–Roch theorem for the Euler characteristic of a divisor D:

$$\chi(D) = h^0(D) - h^1(D) = \deg D + 1 - g,$$

with Serre duality and the adelic interpretation of cohomology.

Theorem 3 (Riemann–Roch theorem). *For every $D \in \operatorname{Div}(X)$ we have*

$$h^0(D) - h^0(K - D) = \deg D + 1 - g.$$

An effective divisor D on X is said to be *non-special* if $h^0(K - D) = 0$. It follows from the Riemann–Roch theorem that an effective divisor D of degree g is non-special if and only if $h^0(D) = 1$. In other words, the only rational functions whose poles are contained in an effective non-special divisor of degree g are the constant functions.

§ 3. Differential and integral calculus

From now on we assume that the algebraically closed field k has characteristic 0 and the algebraic curve X has genus $g \geqslant 1$.

3.1. Differentials of the second kind and 'additive functions'. Following the classical terminology, we call a Kähler differential $\omega \in \Omega^1_{F/k}$ a *differential of the second kind* if $\operatorname{Res}_P \omega = 0$ for all $P \in X$. The k-vector space $\Omega^{(2\mathrm{nd})}$ of differentials of the second kind on X carries a canonical skew-symmetric bilinear form $(\ ,\ )_X$, which is defined as follows. For every $\omega \in \Omega^{(2\mathrm{nd})}$ let $x = \{x_P\}_{P \in X} \in \mathbb{A}_X$ be an adèle satisfying the equality

$$d\, x_P = \omega\big|_P \qquad \forall\, P \in X.$$

For every $P \in X$ there is a unique (up to an additive constant in k) element $x_P \in F_P$ with this property, and we have $x_P \in \mathcal{O}_P$ for all but finitely many $P \in X$. We define $x = d^{-1}\omega$ and put

$$(\omega_1, \omega_2)_X = \sum_{P \in X} \operatorname{Res}_P(d^{-1}\omega_1\, \omega_2), \qquad \omega_1, \omega_2 \in \Omega^{(2\mathrm{nd})}.$$

The bilinear form $(\ ,\ )_X$ is independent of the choice of the additive constants in the definition of d^{-1} and is skew-symmetric. When X is a Riemann surface, the bilinear form $(\ ,\ )_X$ corresponds to the standard pairing in the cohomology under the isomorphism $\Omega^{(2\mathrm{nd})}/dF \simeq H^1_{\mathrm{dR}}(X)$ (see [16], Ch. III, § 5).

The infinite-dimensional k-vector space $\Omega^{(2\mathrm{nd})}$ has a g-dimensional subspace $\Omega^{(1\mathrm{st})} = \Omega^1_{F/k}(0)$ of the differentials of the first kind. The infinite-dimensional subspace $\Omega^{(1\mathrm{st})} \oplus dF$ of $\Omega^{(2\mathrm{nd})}$ is isotropic with respect to the bilinear form $(\ ,\ )_X$. Since there is no canonical choice of the isotropic complementary subspace to $\Omega^{(1\mathrm{st})} \oplus dF$ in $\Omega^{(2\mathrm{nd})}$, the exact sequence

$$0 \to \Omega^{(1\mathrm{st})} \oplus dF \to \Omega^{(2\mathrm{nd})} \to \Omega^{(2\mathrm{nd})}/(\Omega^{(1\mathrm{st})} \oplus dF) \to 0$$

does not split canonically. Nevertheless we have the following fundamental result (see [11], Ch. VI, § 8 and [12], Ch. III, §§ 5.3, 5.4), which may be regarded as an algebraic analogue of de Rham's theorem.

Theorem 4. (i) *The restriction of the bilinear form* $(\ ,\ )_X$ *to* $\Omega^{(2\mathrm{nd})}/dF$ *is non-degenerate and*

$$\dim_k \Omega^{(2\mathrm{nd})}/dF = 2g.$$

(ii) *For every effective non-special divisor* D *on* X *of degree* g *there is an isomorphism*

$$\Omega^{(2\mathrm{nd})}/dF \simeq \Omega^{(2\mathrm{nd})} \cap \Omega^1_{F/k}(2D).$$

(iii) *Let* $D = P_1 + \cdots + P_g$ *be a non-special divisor with distinct points. Then every choice of local uniformizers* t_i *at* P_i *determines a symplectic basis* $\{\theta_i, \omega_i\}_{i=1}^g$ *of the* k*-vector space* $\Omega^{(2\mathrm{nd})} \cap \Omega^1_{F/k}(2D)$ *with respect to the symplectic form* $(\ ,\ )_X$:

$$(\theta_i, \theta_j)_X = (\omega_i, \omega_j)_X = 0, \qquad (\theta_i, \omega_j)_X = \delta_{ij}, \quad i, j = 1, \ldots, g.$$

This basis consists of differentials θ_i *of the first kind and differentials* ω_i *of the second kind which are uniquely determined by the conditions*

$$v_{P_i}(\theta_j - \delta_{ij} dt_i) > 0 \quad \textit{and} \quad v_{P_i}(\omega_j - \delta_{ij} t_i^{-2} dt_i) > 0,$$

where $i, j = 1, \ldots, g$.

(iv) *The subspace* $k \cdot \omega_1 \oplus \cdots \oplus k \cdot \omega_g$ *is an isotropic complement of* $\Omega^{(1\mathrm{st})} \oplus dF$ *in* $\Omega^{(2\mathrm{nd})}$.

Proof. Let $(\omega)_\infty = n_1 Q_1 + \cdots + n_l Q_l$ be the polar divisor of $\omega \in \Omega^{(2\mathrm{nd})}$. Since $\operatorname{char} k = 0$, for every Q_i there is an $f_i \in F$ such that $v_{Q_i}(\omega - df_i) \geqslant 0$. We define $x = \{x_P\}_{P \in X} \in \mathbb{A}_X$ by the formulae

$$x_P = \begin{cases} f_i\big|_{Q_i}, & P = Q_i, \ \ i = 1, \ldots, l, \\ 0, & P \neq Q_i, \ \ i = 1, \ldots, l. \end{cases}$$

Since D is a non-special divisor of degree g, we have $\Omega^1_{F/k}(D) = \{0\}$ and, by Serre duality, $\mathbb{A}_X(D) + F = \mathbb{A}_X$. Thus there is an $f \in F$ with the property $v_P(f - x) \geqslant -v_P(D)$ for all $P \in X$, whence $(\omega - df) \geqslant -2D$. Since D is non-special, such an f is unique. This proves part (ii).

To prove (i), we observe that by the Riemann–Roch theorem,

$$\dim_k \Omega^1_{F/k}(2D) = 3g-1, \qquad \dim_k \Omega^1_{F/k}(D) = 2g-1.$$

Let $\Omega^{(3\mathrm{rd})}$ be the k-vector space of differentials of the third kind. This subspace of $\Omega^1_{F/k}$ is formed by the differentials with only simple poles. Since $\Omega^{(2\mathrm{nd})} \cap \Omega^{(3\mathrm{rd})} = \Omega^{(1\mathrm{st})}$ and $\Omega^{(3\mathrm{rd})} \cap \Omega^1_{F/k}(2D) = \Omega^1_{F/k}(D)$, we conclude that

$$\dim_k \Omega^{(2\mathrm{nd})} \cap \Omega^1_{F/k}(2D) + \dim_k \Omega^1_{F/k}(D) = \dim_k \Omega^1_{F/k}(2D) + \dim_k \Omega^{(1\mathrm{st})}.$$

Using (ii), we have

$$\dim_k \Omega^{(2\mathrm{nd})}/dF = (3g-1) - (2g-1) + g = 2g.$$

To complete the proof, we define a k-linear map

$$L\colon \Omega^{(2\mathrm{nd})} \cap \Omega^1_{F/k}(2D) \to k^{2g}$$

by the formula $L(\omega) = (\alpha_1(\omega), \dots, \alpha_g(\omega), \beta_1(\omega), \dots, \beta_g(\omega))$, where

$$v_{P_i}\big(\omega - (\alpha_i(\omega)t_i^{-2} + \beta_i(\omega)dt_i)\big) > 0, \qquad i = 1, \dots, g.$$

Since D is non-special, L is an injective map and hence an isomorphism. The differentials ω_i and θ_i are obtained by choosing the only non-zero component of L to be $\alpha_i = 1$ and $\beta_i = 1$ respectively. □

Remark 3. The choice of a non-special effective divisor $D = P_1 + \cdots + P_g$ on X with distinct points P_i and uniformizers t_i may be regarded as an algebraic analogue of the choice of a-cycles on a compact Riemann surface of genus $g \geqslant 1$. Correspondingly, the differentials θ_i are analogues of differentials of the first kind with normalized a-periods, and the differentials ω_i are analogues of differentials of the second kind with second-order poles, zero a-periods and normalized b-periods. The symplectic property of the basis $\{\theta_i, \omega_i\}_{i=1}^g$ is an analogue of the reciprocity law for differentials of the first kind and the second kind (see [8], Ch. 5, § 1 and [9], Ch. VI, § 3).

Remark 4. It is not necessary to require all the points of the non-special effective divisor D of degree g to be distinct. Theorem 4 and all the other results in this paper can easily be modified to include divisors with multiple points.

A differential ω of the second kind is said to have zero a-periods if

$$(\omega, \omega_i)_X = 0, \qquad i = 1, \dots, g.$$

It follows from Theorem 4 that a differential of the first kind with zero a-periods is zero. The vector space $\Omega_0^{(2\mathrm{nd})}$ of differentials of the second kind with zero a-periods has the following properties.

Proposition 1. (i) *The k-vector space $\Omega_0^{(2\mathrm{nd})}$ is an isotropic complement of $\Omega^{(1\mathrm{st})}$ in $\Omega^{(2\mathrm{nd})}$ and*

$$\Omega_0^{(2\mathrm{nd})} = k \cdot \omega_1 \oplus \cdots \oplus k \cdot \omega_g \oplus dF.$$

(ii) *For every $P \in X$, the k-vector space $\Omega_0(*P)$ of differentials of the second kind with zero a-periods and the only pole at P has a natural filtration*

$$\{0\} = \Omega_0(P) \subset \Omega_0(2P) \cdots \subset \Omega_0(nP) \subset \cdots,$$
$$\dim_k \Omega_0(nP) = n - 1.$$

(iii) *There is a direct sum decomposition*

$$\Omega_0^{(2\mathrm{nd})} = \bigoplus_{P \in X} \Omega_0(*P).$$

(iv) *Every differential $\omega \in \Omega_0(nP)$ can be written uniquely as*

$$\omega = df + \sum_{i=1}^{g} c_i \omega_i,$$

where $f \in H^0(X, \mathcal{O}_X(D + (n-1)P))$.

Proof. Part (i) follows from Theorem 4 because D is non-special. Since $\dim_k \Omega^1_{F/k}(nP) = n - 1 + g$, part (ii) follows from the decomposition

$$\Omega^1_{F/k}(nP) = \Omega_0(nP) \oplus \Omega^{(1\mathrm{st})}.$$

Part (iii) follows from part (ii) because every differential $\omega \in \Omega_0^{(2\mathrm{nd})}$ can be written uniquely as the sum of its principal parts at the poles. Since the divisor $D = P_1 + \cdots + P_g$ is non-special, we have $h^0(D + (n-1)P) = n$, and part (iv) also follows from Theorem 4. □

Definition 1. A space of *additive multi-valued functions on X* (additive functions for brevity) is a subspace $\mathcal{A}(X) \subset \mathbb{A}_X$ with the following properties.

AF1) $F \subseteq \mathcal{A}(X)$.

AF2) If $a \in \mathcal{A}(X)$, then $da = \omega \in \Omega^1_{F/k}$ (and hence $\omega \in \Omega^{(2\mathrm{nd})}$).

AF3) If $a \in \mathcal{A}(X)$ and $da = 0$, then $a = c \in k$.

Remark 5. For every differential $\omega \in \Omega^{(2\mathrm{nd})}$, the corresponding adèle $a = \{a_P\}_{P \in X} = d^{-1}\omega$ is determined uniquely up to the choice of additive constants for every $P \in X$. Condition AF3) guarantees that for all $f \in F$ these constants are compatible with the equation $f = d^{-1}(df) + c$, where $c \in k$.

Example 1. Given any non-special effective divisor $D = P_1 + \cdots + P_g$ of degree g on X with distinct points P_i and any choice of the local uniformizers t_i at P_i, we have the following space $\mathcal{A}(X; D)$ of additive functions with zero a-periods. Let $\eta_i \in \mathbb{A}_X$ be solutions of the equations

$$d\eta_i = \omega_i, \qquad i = 1, \ldots, g,$$

with any fixed choice of the additive constants at all points $P \in X$. Since the divisor D is non-special, the subspaces $k \cdot \eta_1 \oplus \cdots \oplus k \cdot \eta_g$ and F of the k-vector space $\mathbb{A}_X$ have zero intersection. Their direct sum

$$\mathcal{A}(X; D) = k \cdot \eta_1 \oplus \cdots \oplus k \cdot \eta_g \oplus F \subset \mathbb{A}_X \tag{6}$$

possesses properties AF1)–AF3) and the map $d\colon \mathcal{A}(X;D) \to \Omega_0^{(2\mathrm{nd})}$ is surjective. Indeed, by Proposition 1 every differential $\omega \in \Omega_0^{(2\mathrm{nd})}$ can be written uniquely in the form

$$\omega = df + \sum_{i=1}^{g} c_i \omega_i, \tag{7}$$

whence

$$a = d^{-1}\omega = f + \sum_{i=1}^{g} c_i \eta_i + c \in \mathcal{A}(X;D). \tag{8}$$

Remark 6. The additive functions $a = d^{-1}\omega \in \mathcal{A}(X,D)$ are algebraic analogues of Abelian integrals of the second kind with zero a-periods on a compact Riemann surface of genus g (see, for example, [8], Ch. V, § 2). We can define

$$\int_P^Q \omega = a(Q) - a(P),$$

where $a(P) = a_P \mod \mathfrak{p} \in k$ for all $P \in X$.

It is remarkable that using the additive functions in Example 1, one can naturally define the uniformizers t_P at all points $P \in X$. They are uniquely determined by the following data: a choice of a non-special divisor $D = P_1 + \cdots + P_g$ with distinct points, uniformizers t_i at P_i and additive functions $\eta_1, \ldots, \eta_g$. For every $P \in X$ let $\omega_P^{(2)} \in \Omega_0(2P)$ be the unique differential of the second kind with the only second-order pole at P and zero a-periods such that

$$\sum_{i=1}^{g} \bigl(\theta_i, \omega_P^{(2)}\bigr)_X = 1. \tag{9}$$

In particular, $\omega_{P_i}^{(2)} = \omega_i$ for $i = 1, \ldots, g$. Let $\eta_P = d^{-1}\omega_P^{(2)} \in \mathcal{A}(X;D)$ be an additive function with the only simple pole at $P \in X$. By (8), η_P is uniquely determined up to an overall additive constant. We fix this constant by requiring that the sum of the constant terms of $\eta_P\big|_{P_i} \in k((t_i))$ over all $i = 1, \ldots, g$ be equal to zero. In particular, $\eta_{P_i} = \eta_i + c_i$ for some $c_i \in k$. For every $P \in X$ we now define the uniformizer t_P by the formula

$$t_P = -\frac{1}{\eta_P}\bigg|_P,$$

and for $\omega_P^{(2)} = d\eta_P$ we have

$$\omega_P^{(2)}\big|_P = t_P^{-2}\, dt_P, \qquad P \in X.$$

Extending this construction, we now endow the subspace $\Omega_0(*P)$ for every $P \in X$ with a basis $\{\omega_P^{(n+1)}\}_{n=1}^{\infty}$ consisting of differentials of the second kind with the only pole at P of order $n+1$ and zero a-periods, where the differentials $\omega_P^{(2)}$ are already specified by (9). Let $\eta_P^{(n)} = d^{-1}\omega_P^{(n+1)} \in \mathcal{A}(X;D)$ be an additive function with the

only pole at $P \in X$ of order n and with the following choice of the overall additive constant in (8). We put $\eta_P^{(1)} = \eta_P$ and require the constant term of $\eta_P^{(n)}\big|_P \in k((t_P))$ to be equal to zero for all $\eta_P^{(n)}$ with $n > 1$. For every $P \in X$ let $\mathcal{A}_P(X, D)$ be the k-linear span of $\eta_P^{(n)}$, $n \in \mathbb{N}$. Then we have a decomposition

$$\mathcal{A}(X, D) = \left(\bigoplus_{P \in X} \mathcal{A}_P(X, D) \right) \oplus k. \tag{10}$$

One can restate the property of isotropy of the subspace $\Omega_0^{(2\mathrm{nd})} = d\mathcal{A}(X; D)$ and the condition AF3) in the following way.

Lemma 1. (i) *For all $P, Q \in X$ and $m, n \in \mathbb{N}$ we have*

$$\mathrm{Res}_P(\eta_P^{(m)} d\eta_Q^{(n)}) = \mathrm{Res}_Q(\eta_Q^{(n)} d\eta_P^{(m)}).$$

(ii) *Every rational function $f \in F$ has a unique 'partial-fraction expansion'*

$$f = \sum_{i=1}^{l} \sum_{j=1}^{n_i} c_{ij} \eta_{Q_i}^{(j)} + c,$$

where $n_1 Q_1 + \cdots + n_l Q_l = (f)_\infty$ is the polar divisor of f and $c, c_{ij} \in k$.

Proof. Since $\mathrm{Res}_Q(da) = 0$ for all $a \in \mathbb{A}_X$, we get, for $P \neq Q$,

$$\begin{aligned} 0 = (\omega_P^{(m+1)}, \omega_Q^{(n+1)})_X &= \mathrm{Res}_P(\eta_P^{(m)} d\eta_Q^{(n)}) + \mathrm{Res}_Q(\eta_P^{(m)} d\eta_Q^{(n)}) \\ &= \mathrm{Res}_P(\eta_P^{(m)} d\eta_Q^{(n)}) - \mathrm{Res}_Q(\eta_Q^{(n)} d\eta_P^{(m)}). \end{aligned}$$

For $P = Q$ we have $0 = (\omega_P^{(m+1)}, \omega_P^{(n+1)})_X = \mathrm{Res}_P(\eta_P^{(m)} d\eta_P^{(n)})$ for all $m, n \in \mathbb{N}$.

Part (ii) follows directly from AF3) since there are $c_{ij} \in k$ such that

$$df - \sum_{i=1}^{l} \sum_{j=1}^{n_i} c_{ij} \omega_{Q_i}^{(j+1)} \in \Omega_0^{(2\mathrm{nd})} \cap \Omega^{(1\mathrm{st})} = \{0\}. \qquad \square$$

Remark 7. Part (i) of Lemma 1 is an algebraic analogue of the classical reciprocity law for differentials of the second kind with zero a-periods on a compact Riemann surface (see, for example, [8], Ch. V, § 1 and [11], Ch. VI, § 3).

Remark 8. In the genus zero case $X = \mathbb{P}_k^1 = k \cup \{\infty\}$ we have $F = k(z)$ and

$$\omega_P^{(n+1)} = \begin{cases} \dfrac{dz}{(z - P)^{n+1}} & \text{for} \quad P \in k, \\ -z^{n-1}\, dz & \text{for} \quad P = \infty. \end{cases}$$

Correspondingly,

$$\eta_P^{(n)}(z) = \begin{cases} -\dfrac{1}{n(z - P)^n} & \text{for} \quad P \in k, \\ -\dfrac{z^n}{n} & \text{for} \quad P = \infty. \end{cases}$$

Remark 9. Put $\mathbb{O}_X = \mathbb{A}_X(0) = \prod_{P\in X} \mathcal{O}_X$. By Lemma 1 we have

$$\mathbb{A}_X = \mathcal{A}(X, D_{\mathrm{ns}}) + \mathbb{O}_X$$

while Serre's adelic interpretation of cohomology yields

$$\mathbb{A}_X/(F + \mathbb{O}_X) = H^1(X, \mathcal{O}_X).$$

Remark 10. The condition that the field k of constants is algebraically closed is not necessary: all results in this section remain valid for any field of constants of characteristic 0 if we replace the field k by the residue class field $k(P) = \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}$ and use the trace map $\mathrm{Tr}_{k(P)/k}\colon k(P) \to k$. For example, for the bilinear form $(\ ,\ )_X$ we have

$$(\omega_1, \omega_2)_X = \sum_{P\in X} \mathrm{Tr}_{k(P)/k} \operatorname{Res}_P(d^{-1}\omega_1\,\omega_2).$$

Remark 11. The multiplicative analogue of the k-vector space $\mathcal{A}(X)$ of additive multi-valued functions is the group $\mathcal{M}(X)$ of multiplicative multi-valued functions on X. This subgroup of the group of invertible elements of the adèle ring $\mathbb{A}_X$ is defined by the following properties. It contains F^* as a subgroup, we have $d\log m = m^{-1}dm = \omega \in \Omega^1_{F/k}$ for all $m \in \mathcal{M}(X)$, and if $m \in \mathcal{M}(X)$ satisfies $d\log m = 0$, then $m = c \in k^*$. It also seems natural to assume (as was done in a preliminary version of this paper) that the following multiplicative analogue of Lemma 1 holds. Every rational function $f \in F^*$ can be written uniquely as a product of multiplicative multi-valued functions with one zero and one pole obeying the natural generalized Weil reciprocity law on X (see [1], [17]). However, the referee pointed out that this assertion contradicts the non-triviality of Poincaré's bi-extension over the square of the Jacobian of X [18].

§ 4. The local theory

Let K be a complete closed field, that is, a complete discrete valuation field with valuation ring $\mathcal{O}_K$, maximal ideal $\mathfrak{p}$ and algebraically closed residue field $k = \mathcal{O}_K/\mathfrak{p}$. Every local uniformizer t determines an isomorphism $K \simeq k((t))$. Therefore K may be interpreted as a 'geometric loop algebra' over k. The main example of a complete closed field is $K = F_P$, where P is a point on an algebraic curve X over k.

Here we describe some infinite-dimensional Lie algebras naturally associated with K and construct their irreducible highest-weight modules. When $K = F_P$, these objects determine local quantum field theories at $P \in X$. Specifically, we consider the following local quantum field theories (QFT):

1) the 'QFT of additive bosons', which corresponds to the Heisenberg Lie algebra $\mathfrak{g}$ (a one-dimensional central extension of the geometric loop algebra $\mathfrak{gl}_1(K) = K$),

2) the 'QFT of charged bosons', which corresponds to the lattice Lie algebra $\mathfrak{l}$ associated with the Heisenberg Lie algebra $\mathfrak{g}$ and the lattice $\mathbb{Z}$.

4.1. The Heisenberg algebra. Let $\Omega^1_{K/k}$ be the K-module of Kähler differentials. We put $\widetilde{\Omega}^1_{K/k} = \Omega^1_{K/k}/\mathcal{Q}$, where $\mathcal{Q} = \bigcap_{n\geqslant 0} \mathfrak{p}^n d(\mathcal{O})$ (see § 2.2). The

Abelian Lie algebra $\mathfrak{gl}_1(K) = K$ over the field k is endowed with a natural bilinear skew-symmetric form $c\colon \wedge^2 K \to k$ by the formula

$$c(f,g) = -\operatorname{Res}(f\,dg), \qquad f,g \in K,$$

where $dg \in \widetilde{\Omega}^1_{K/k}$. The bilinear form c is continuous with respect to the $\mathfrak{p}$-adic topology on K and the discrete topology on k. Hence $c \in H^2_c(K,k) \simeq \operatorname{Hom}_c(\wedge^2 K, k)$, where $\operatorname{Hom}_c(\wedge^2 K, k)$ is the group of continuous 2-cocycles on K with values in k.

Definition 2. The *Heisenberg Lie algebra* $\mathfrak{g}$ is the one-dimensional central extension of K

$$0 \to k \cdot C \to \mathfrak{g} \to K \to 0$$

with the 2-cocycle c.

Writing $[\,,\,]$ for the Lie bracket in $\mathfrak{g} = K \oplus k \cdot C$, we have

$$[f + a\,C, g + b\,C] = c(f,g)\,C, \qquad f,g \in K, \quad a,b \in k.$$

The Lie subalgebra $\mathfrak{g}_+ = \mathcal{O}_K \oplus k \cdot C$ is a maximal Abelian subalgebra of $\mathfrak{g}$.

Remark 12. Let $\operatorname{Aut}\mathcal{O} = \{u \in \mathcal{O}\colon v(u) = 1\}$ be the group of continuous automorphisms of the valuation ring $\mathcal{O} = k[[t]]$ (see [14]). One can easily show that every continuous linear map $l\colon k((t)) \otimes_k k((t)) \to k$ which satisfies

$$l(f \circ u, g \circ u) = l(f,g)$$

for all $f,g \in k((t))$ and $u \in \operatorname{Aut}\mathcal{O}$ is a constant multiple of c. This explains the natural role of the 2-cocycle c of K. In particular, every $\operatorname{Aut}\mathcal{O}$-invariant bilinear form l is necessarily skew-symmetric. This may be regarded as a simple algebraic analogue of the spin-statistics theorem.

Definition 3. A *module of the Heisenberg algebra* $\mathfrak{g}$ is a k-vector space V with the discrete topology and with a k-algebra homomorphism $\rho\colon \mathfrak{g} \to \operatorname{End} V$ such that the corresponding action of $\mathfrak{g}$ on V is continuous and $\rho(C) = \mathbf{I}$ is the identity endomorphism of V.

Equivalently, for every $v \in V$ there is an open subspace U of K which is commensurable with $\mathfrak{p}$ and annihilates v: $\rho(U)v = 0$. Putting $\mathbf{f} = \rho(f) \in \operatorname{End} V$ for all $f \in K$, we have

$$[\mathbf{f}, \mathbf{g}] = c(f,g)\mathbf{I}$$

and thus obtain a projective representation of the Abelian Lie algebra K.

Remark 13. Any choice of the uniformizer t for K determines an isomorphism $K \simeq k((t))$ and a basis $\{t^n\}_{n\in\mathbb{Z}}$ in K. Putting $\boldsymbol{\alpha}_n = \rho(t^n)$ and using the formulae $c(t^m, t^n) = m\delta_{m,-n}$, we get the commutation relations of the 'oscillator algebra',

$$[\boldsymbol{\alpha}_m, \boldsymbol{\alpha}_n] = m\delta_{m,-n}\mathbf{I}.$$

They characterize free bosons in the two-dimensional QFT.

Definition 4. An *irreducible highest-weight module of the Heisenberg algebra* $\mathfrak{g}$ is an irreducible $\mathfrak{g}$-module with a vector $\mathbf{1} \in V$ which is annihilated by the Abelian subalgebra $\mathcal{O}_K \oplus \{0\}$.

The following result is well known (see, for example, [13], Lemma 9.13).

Theorem 5. *Each irreducible highest-weight module of the Heisenberg Lie algebra* $\mathfrak{g}$ *is either the trivial one-dimensional module* $k = k \cdot \mathbf{1}$ *with highest vector* $\mathbf{1} = 1 \in k$, *or the Fock module*

$$\mathscr{F} = \operatorname{ind}_{\mathfrak{g}_+}^{\mathfrak{g}} k$$

induced from the one-dimensional $\mathfrak{g}_+$*-module* k.

Remark 14. Let $U\mathfrak{g}$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. By definition,

$$\mathscr{F} = U\mathfrak{g} \otimes_{U\mathfrak{g}_+} k,$$

where $U\mathfrak{g}$ is regarded as a right $U\mathfrak{g}_+$-module. Equivalently,

$$\mathscr{F} = \mathscr{W}/\mathscr{D}, \tag{11}$$

where $\mathscr{W}$ is the Weyl algebra of $\mathfrak{g}$, that is, the quotient of $U\mathfrak{g}$ by the ideal generated by $C - \mathbf{1}$ (with $\mathbf{1}$ now standing for the identity in $U\mathfrak{g}$) and $\mathscr{D}$ is the left ideal in $\mathscr{W}$ generated by $\mathcal{O}_K \oplus \{0\}$.

An explicit realization of the Fock module $\mathscr{F}$ (the bosonic Fock space) depends on a decomposition of K into a direct sum of subspaces isotropic with respect to the bilinear form c:

$$K = K_+ \oplus K_-, \tag{12}$$

where the subspace $K_+ = \mathcal{O}_K$ is defined canonically. In this case,

$$\mathscr{F} \simeq \operatorname{Sym}^\bullet K_- \tag{13}$$

is the symmetric algebra of the k-vector space K_-. The Fock space $\mathscr{F}$ is a $\mathbb{Z}$-graded commutative algebra

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{F}^{(n)},$$

where $\mathscr{F}^{(n)} \simeq \operatorname{Sym}^n K_-$, $\mathscr{F}^{(0)} = k \cdot \mathbf{1}$ and $\mathscr{F}^{(n)} = \{0\}$ for $n < 0$. For every $f = f_+ + f_- \in K$, the operator $\mathbf{f} = \rho(f) \in \operatorname{End} \mathscr{F}$ is defined by the formula

$$\mathbf{f} \cdot v = f_- \odot v + \sum_{i=1}^{k} c(f, v_i)\, v^i = f_- \odot v - \sum_{i=1}^{k} \operatorname{Res}(f_+\, dv_i)\, v^i, \tag{14}$$

where $v = v_1 \odot \cdots \odot v_k \in \mathscr{F}^{(k)}$ and $v^i = v_1 \odot \cdots \odot \widehat{v_i} \odot \cdots \odot v_k \in \mathscr{F}^{(k-1)}$, $i = 1, \ldots, k$. Here $\odot$ stands for the multiplication in $\operatorname{Sym}^\bullet K_-$. In particular,

$$\mathbf{f} \cdot \mathbf{1} = f_-.$$

The Fock space $\mathscr{F}$ is endowed with the linear topology given by the filtration associated with the $\mathbb{Z}$-grading and independent of the decomposition (12).

Remark 15. Any choice of the uniformizer t determines an isomorphism $K \simeq k((t))$, and one can take $K_- = t^{-1}k[t^{-1}]$. The map

$$\mathscr{F}^{(n)} \ni v = t^{-m_1} \odot \cdots \odot t^{-m_n} \mapsto x_{m_1} \cdots x_{m_n} \in k[x_1, x_2, \dots]$$

determines an isomorphism $\mathscr{F} \simeq k[x_1, x_2, \dots]$ between the bosonic Fock space and the polynomial ring in infinitely many variables $\{x_n\}_{n\in\mathbb{N}}$. Under this map we have $\boldsymbol{\alpha}_n \mapsto n\partial/\partial x_n$, $\boldsymbol{\alpha}_{-n} \mapsto x_n$, $n > 0$ (the operator of multiplication by x_n), and $\boldsymbol{\alpha}_0 \mapsto 0$.

Remark 16. For an arbitrary complete closed field K there is no canonical choice of the isotropic subspace K_- complementary to $K_+ = \mathcal{O}_K$. However, any choice of an effective non-special divisor $D = P_1 + \cdots + P_g$ of degree g on the algebraic curve X and uniformizers t_i at P_i determines such isotropic subspaces K_- for all fields $K = F_P$, $P \in X$. Namely, let $\mathcal{A}(X, D)$ be the k-vector space of additive functions defined in Example 1, and let $\mathcal{A}_P(X, D)$ be the subspace of additive functions with the only pole at P. We put

$$K_- = \mathcal{A}_P(X, D)\big|_P \subset K.$$

By part (i) of Lemma 1, the subspace K_- is isotropic with respect to c and we have the decomposition (12). The subspace K_- is spanned by the elements $v_P^{(n)} = \eta_P^{(n)}\big|_P$, $n \in \mathbb{N}$, and $dK_- = \Omega_0(*P)\big|_P$.

The bilinear form c has the one-dimensional kernel k. Since $\mathcal{O}_K/k = \mathfrak{p}$, the form c determines a continuous non-degenerate pairing $c\colon \mathfrak{p} \otimes K_- \to k$, whence $\mathfrak{p} = K_-^\vee = \operatorname{Hom}(K_-, k)$ is the topological dual of the k-vector space K_-. The topological dual of the bosonic Fock space $\mathscr{F}$ is accordingly equal to the k-vector space $\mathscr{F}^\vee = \overline{\operatorname{Sym}^\bullet \mathfrak{p}}$ which is the completion of $\operatorname{Sym}^\bullet \mathfrak{p}$ with respect to the linear topology given by the filtration $\{F^n \operatorname{Sym}^\bullet \mathfrak{p}\}_{n=0}^\infty$,

$$F^n \operatorname{Sym}^\bullet \mathfrak{p} = \bigoplus_{i=0}^{n} \operatorname{Sym}^i \mathfrak{p}.$$

The continuous pairing $(\ ,\ )\colon \mathscr{F}^\vee \otimes \mathscr{F} \to k$ is uniquely determined by the pairing between $\operatorname{Sym}^\bullet \mathfrak{p}$ and $\mathscr{F} = \operatorname{Sym}^\bullet K_-$ and is defined recursively by the formula

$$(u, v) = \delta_{kl} \sum_{i=1}^{l} c(u_1, v_i)(u^1, v^i), \tag{15}$$

where $u = u_1 \odot \cdots \odot u_k = u_1 \odot u^1 \in \operatorname{Sym}^k \mathfrak{p}$ and $v = v_1 \odot \cdots \odot v_l = v_i \odot v^i \in \mathscr{F}^{(l)}$. The dual bosonic Fock space $\mathscr{F}^\vee$ is a right $\mathfrak{g}$-module with lowest-weight vector $\mathbf{1}^\vee$ annihilated by the subspace $K_- \oplus k$.

The representation ρ of the Heisenberg algebra $\mathfrak{g}$ on $\mathscr{F}$ determines a contragradient representation $\rho^\vee$ of the Heisenberg algebra $\mathfrak{g}$ on $\mathscr{F}^\vee$ by the formula

$$(u \cdot \rho^\vee(f), v) = (u, \rho(f) \cdot v) \qquad \forall\, u \in \mathscr{F}^\vee,\ v \in \mathscr{F}.$$

More explicitly, put $f = \widetilde{f}_+ + \widetilde{f}_- \in K$, where now $\widetilde{f}_+ \in \mathfrak{p}$ and $\widetilde{f}_- \in K_- \oplus k$. Then it follows from (14) and (15) that the operator $\mathbf{f} = \rho^\vee(f) \in \operatorname{End} \mathscr{F}^\vee$ is given by

$$u \cdot \mathbf{f} = \widetilde{f}_+ \odot u + \sum_{i=1}^{k} c(u_i, f) u^i = \widetilde{f}_+ \odot u + \sum_{i=1}^{k} \operatorname{Res}(\widetilde{f}_- du_i) u^i, \tag{16}$$

where $u = u_1 \odot \cdots \odot u_k \in \operatorname{Sym}^k \mathfrak{p}$ and $u^i = u_1 \odot \cdots \odot \widehat{u}_i \odot \cdots \odot u_k \in \operatorname{Sym}^{k-1} \mathfrak{p}$.

4.2. The lattice algebra. Let $k[\mathbb{Z}]$ be the group algebra of the additive group $\mathbb{Z}$. As a k-vector space, $k[\mathbb{Z}]$ has a basis $\{e_n\}_{n\in\mathbb{Z}}$, $e_m e_n = e_{m+n}$. For every decomposition (12) we define the 'constant term' of any $f \in K$ as $f(0) = f_+ \mod \mathfrak{p} \in k$. Hence we have $f(0) = 0$ for $f \in K_-$.

Remark 17. If $K = F_P$ and $K_- = \mathcal{A}_P(X, D)\big|_P$, then $f(0)$ is the constant term of the formal Laurent expansion of $f \in k((t_P))$ with respect to the uniformizer t_P of K, defined in § 3.1.

Definition 5. The lattice Lie algebra $\mathfrak{l}$ associated with the decomposition (12) is a semidirect sum of the Heisenberg Lie algebra $\mathfrak{g}$ and the Abelian Lie algebra $k[\mathbb{Z}]$ with the Lie bracket

$$[f + aC + \alpha e_m, g + bC + \beta e_n] = c(f, g)C + m\alpha g(0)e_m - n\beta f(0)e_n,$$

where $f + aC$, $g + bC \in \mathfrak{g}$.

The corresponding irreducible highest-weight module $\mathscr{B}$ of $\mathfrak{l}$ is given by

$$\mathscr{B} = k[\mathbb{Z}] \otimes \mathscr{F},$$

where $k[\mathbb{Z}]$ acts by multiplication and K acts by the formula

$$\mathbf{f}(e_n \otimes v) = -nf(0)e_n \otimes v + e_n \otimes (\mathbf{f} \cdot v), \qquad f \in K, \quad v \in \mathscr{F}.$$

The module $\mathscr{B}$ (the Fock space of 'charged bosons') is a $\mathbb{Z}$-graded commutative algebra,

$$\mathscr{B} = \bigoplus_{n\in\mathbb{Z}} \mathscr{B}^{(n)}, \qquad \mathscr{B}^{(n)} = k \cdot e_n \otimes \mathscr{F}.$$

The elements e_n, $n \in \mathbb{Z}$, correspond to the shift operators $\mathbf{e}_n = \mathbf{e}^n$ in $\mathscr{B}$, where

$$\mathbf{e}(e_n \otimes v) = e_{n+1} \otimes v, \qquad v \in \mathscr{F}.$$

Remark 18. Using the canonical isomorphism $K^*/\mathcal{O}_K^* \simeq \mathbb{Z}$ induced by the valuation $v\colon K^* \to \mathbb{Z}$, one can also define the Fock space $\mathscr{B}$ as the space of all functions

$$F\colon K^*/\mathcal{O}_K^* \to \mathscr{F}$$

with finite support.

Remark 19. For every choice of uniformizer t for K, the map

$$\mathscr{B}^{(n)} \ni e_n \otimes (t^{-m_1} \odot \cdots \odot t^{-m_l}) \mapsto e^{nx_0} x_{m_1} \cdots x_{m_l} \in e^{nx_0} k[x_1, x_2, \dots]$$

establishes an isomorphism $\mathscr{B} \simeq k[e^{x_0}, e^{-x_0}, x_1, x_2, \dots]$. Under this map we have $\boldsymbol{\alpha}_n \mapsto n\partial/\partial x_n$, $\boldsymbol{\alpha}_{-n} \mapsto x_n$, $n > 0$, $\boldsymbol{\alpha}_0 \mapsto -\partial/\partial x_0$ and $\mathbf{e} \mapsto e^{x_0}$ (the operator of multiplication by e^{x_0}).

The topological dual of $\mathscr{B}$ is the k-vector space

$$\mathscr{B}^\vee = \bigoplus_{n\in\mathbb{Z}} k \cdot q^n \otimes \mathscr{F}^\vee,$$

where $\{q^n\}_{n\in\mathbb{Z}}$ is the basis in $k[\mathbb{Z}]^\vee$ dual to the basis $\{e_n\}_{n\in\mathbb{Z}}$. The continuous pairing $(\ ,\ )\colon \mathscr{B}^\vee \otimes \mathscr{B} \to k$ is given by

$$(q^m \otimes u, e_n \otimes v) = (u, v)\delta_{mn}, \qquad u \in \mathscr{F}^\vee, \quad v \in \mathscr{F}.$$

As for the Heisenberg algebra, the representation ρ of the lattice algebra $\mathfrak{l}$ on $\mathscr{B}$ determines a contragradient representation $\rho^\vee$ on $\mathscr{B}^\vee$. The dual Fock space $\mathscr{B}^\vee$ is a right $\mathfrak{l}$-module with lowest-weight vector $\mathbf{1}^\vee$ annihilated by K_-.

§ 5. The global theory

Given an algebraic curve X over an algebraically closed field k of characteristic 0, we shall define the global versions of the local QFT's introduced in § 4. One can briefly characterize these global QFT's as follows.

1. The 'QFT of additive bosons on X' corresponds to the global Heisenberg algebra $\mathfrak{g}_X$ (the restricted direct sum of the local Heisenberg algebras $\mathfrak{g}_P$ over all points $P \in X$). The global Fock space $\mathscr{F}_X$ is defined as the restricted tensor product of the local Fock spaces $\mathscr{F}_P$ over all points $P \in X$. The global Fock space $\mathscr{F}_X$ is a highest-weight $\mathfrak{g}_X$-module. There is a linear functional $\langle\,\cdot\,\rangle\colon \mathscr{F}_X \to k$ (the expectation value functional) which is uniquely determined by the properties of normalization and invariance with respect to the space of additive functions.

2. The 'QFT of charged bosons on X' corresponds to the global lattice algebra $\mathfrak{l}_X$. The global charged Fock space $\mathscr{B}_X$ is a highest-weight $\mathfrak{l}_X$-module and there is a unique expectation value functional $\langle\,\cdot\,\rangle\colon \mathscr{B}_X \to k$ with similar properties.

5.1. Additive bosons on X. The QFT of additive bosons consists of the following data.

AB1) An effective non-special divisor $D_{\mathrm{ns}} = P_1 + \cdots + P_g$ of degree g on X with distinct points, uniformizers t_i at P_i and the k-vector space $\mathcal{A}(X, D_{\mathrm{ns}})$ of additive functions (a subspace of $\mathbb{A}_X$ containing $F = k(X)$) introduced in Example 1.

AB2) The local QFT's of additive bosons (the highest-weight $\mathfrak{g}_P$-modules $\mathscr{F}_P$ for all points $P \in X$).

AB3) The global Heisenberg algebra $\mathfrak{g}_X$ (the one-dimensional central extension of the Abelian Lie algebra $\mathfrak{gl}_1(\mathbb{A}_X) = \mathbb{A}_X$ by the cocycle $c_X = \sum_{P\in X} c_P$).

AB4) A highest-weight $\mathfrak{g}_X$-module (the global Fock space $\mathscr{F}_X$, which is the restricted tensor product of $\mathscr{F}_P$ over all points $P \in X$).

AB5) An expectation value functional, that is, a linear map $\langle\cdot\rangle\colon \mathscr{F}_X \to k$ with the following properties:

(i) $\langle \mathbf{1}_X\rangle = 1$, where $\mathbf{1}_X \in \mathscr{F}_X$ is the highest-weight vector,

(ii) $\langle \mathbf{a}\cdot v\rangle = 0$ for all $a \in \mathcal{A}(X, D_{\mathrm{ns}})$ and $v \in \mathscr{F}_X$.

The data AB1) and AB2) have already been described in §§ 3.1 and 4.1. Here we introduce the global Heisenberg algebra $\mathfrak{g}_X$, construct the corresponding global Fock space $\mathscr{F}_X$ and prove that there is a unique expectation value functional $\langle\cdot\rangle$ with properties (i) and (ii).

Let $c_X\colon \mathbb{A}_X \times \mathbb{A}_X \to k$ be the global bilinear form

$$c_X(x,y) = \sum_{P\in X} c_P(x_P, y_P) = -\sum_{P\in X} \operatorname{Res}_P(x_P\, dy_P), \qquad x, y \in \mathbb{A}_X.$$

Definition 6. The *global Heisenberg Lie algebra* $\mathfrak{g}_X$ is the one-dimensional central extension of the Abelian Lie algebra $\mathbb{A}_X$

$$0 \to k\,C \to \mathfrak{g}_X \to \mathbb{A}_X \to 0$$

by the 2-cocycle c_X.

The Lie subalgebra $\mathfrak{g}_X^+ = \mathbb{O}_X \oplus kC$ is the maximal Abelian subalgebra of $\mathfrak{g}_X$.

Definition 7. The *global Fock space* $\mathscr{F}_X$ is an irreducible $\mathfrak{g}_X$-module with vector $\mathbf{1}_X$ annihilated by the Abelian subalgebra $\mathbb{O}_X \oplus \{0\}$.

As in the local case, the global Fock module is induced from the one-dimensional $\mathfrak{g}_X^+$-module:

$$\mathscr{F}_X = \operatorname{ind}_{\mathfrak{g}_X^+}^{\mathfrak{g}_X} k.$$

By what was said in § 4, we have a decomposition (12) for $K = F_P$, $P \in X$, where $F_P^{(+)} = \mathcal{O}_P$ and $F_P^{(-)} = \mathcal{A}_P(X, D)\big|_P$. This yields the following decomposition of the k-vector space $\mathbb{A}_X$ into a direct sum of subspaces isotropic with respect to c_X:

$$\mathbb{A}_X = \mathbb{O}_X \oplus \mathcal{F}_X^{(-)}. \tag{17}$$

Here

$$\mathcal{F}_X^{(-)} = \coprod_{P\in X} F_P^{(-)}$$

is the restricted direct product over all $P \in X$ with respect to the zero subspaces $\{0\} \subset F_P^{(-)}$. The decomposition (17) gives rise to an isomorphism

$$\mathscr{F}_X \simeq \operatorname{Sym}^\bullet \mathcal{F}_X^{(-)}.$$

The global Fock space $\mathscr{F}_X$ carries a linear topology given by the natural filtration associated with the $\mathbb{Z}$-grading.

Equivalently, $\mathscr{F}_X$ may be defined as the tensor product

$$\mathscr{F}_X = \widehat{\bigotimes_{P\in X}} \mathscr{F}_P,$$

which is restricted with respect to the vectors $\mathbf{1}_P \in \mathscr{F}_P$ and is endowed with the product topology. In other words, $\mathbf{1}_X = \bigotimes_{P\in X} \mathbf{1}_P$, and $\mathscr{F}_X$ is spanned by the vectors

$$v = \bigotimes_{P\in X} v_P,$$

where $v_P = \mathbf{1}_P$ for all but finitely many $P \in X$. For every $P \in X$ we have $v = v_P \otimes v^P$, where $v^P = \bigotimes_{Q\in X} \widetilde{v}_Q$, $\widetilde{v}_Q = v_Q$ for $Q \neq P$ and $\widetilde{v}_P = \mathbf{1}_P$. We denote the corresponding representation of $\mathfrak{g}_P$ on $\mathscr{F}_P$, $P \in X$, by ρ_P, and the representation of $\mathfrak{g}_X$ on $\mathscr{F}_X$ by ρ. Putting $\mathbf{x} = \rho(x) \in \operatorname{End}\mathscr{F}_X$ for $x = \{x_P\}_{P\in X} \in \mathbb{A}_X$ and taking any $v = \bigotimes_{P\in X} v_P$, we have

$$\mathbf{x}\cdot v = \sum_{P\in X} \mathbf{x}_P \cdot v_P \otimes v^P,$$

where $\mathbf{x}_P = \rho_P(x_P) \in \operatorname{End}\mathscr{F}_P$.

Put

$$\mathfrak{P}_X = \prod_{P\in X} \mathfrak{p}.$$

The topological dual of the global Fock space $\mathscr{F}_X$ is the k-vector space $\mathscr{F}_X^\vee = \overline{\operatorname{Sym}^\bullet \mathfrak{P}_X}$, which is the completion of $\operatorname{Sym}^\bullet \mathfrak{P}_X$ with respect to the linear topology given by the natural filtration associated with the $\mathbb{Z}$-grading. The dual global Fock space $\mathscr{F}_X^\vee$ is a right $\mathfrak{g}_X$-module with lowest-weight vector $\mathbf{1}_X^\vee$ annihilated by the Abelian subalgebra $\mathcal{F}_X^{(-)} \oplus \{0\}$. Equivalently,

$$\mathscr{F}_X^\vee = \widehat{\bigotimes_{P\in X}} \mathscr{F}_P^\vee$$

is the completion of the tensor product restricted with respect to the vectors $\mathbf{1}_P^\vee$. The completion is taken with respect to the double filtration $\{F^{mn} \operatorname{Sym}^\bullet \mathfrak{P}_X\}$, where

$$F^{mn} \operatorname{Sym}^\bullet \mathfrak{P}_X = \sum_{i=0}^{m} \sum_{P_1,\dots,P_i\in X} \Biggl(\bigoplus_{l_1+\dots+l_i=0}^{n} \operatorname{Sym}^{l_1} \mathfrak{p}_1 \otimes \dots \otimes \operatorname{Sym}^{l_i} \mathfrak{p}_i \Biggr).$$

In other words, the elements of $\mathscr{F}_X^\vee$ are infinite formal sums

$$u = \sum_{n=0}^{\infty} \sum_{P_1,\dots,P_n\in X} a_{P_1\dots P_n} u_{P_1\dots P_n},$$

where the $u_{P_1\dots P_n}$ belong to the completion $\overline{\mathscr{F}^\vee}_{P_1\dots P_n}$ of the tensor product

$$\mathscr{F}^\vee_{P_1\dots P_n} = \mathscr{F}^\vee_{P_1} \otimes \dots \otimes \mathscr{F}^\vee_{P_n}$$

with respect to the filtration

$$F^m \mathscr{F}^\vee_{P_1\dots P_n} = \bigoplus_{l_1+\dots+l_n=0}^{m} \Bigl(\operatorname{Sym}^{l_1} \mathfrak{p}_1 \otimes \dots \otimes \operatorname{Sym}^{l_n} \mathfrak{p}_n \Bigr).$$

Let $\{u_P^{(n)}\}_{n\in\mathbb{N}}$ be the basis of $\mathfrak{p}$ dual to the basis $\{v_P^{(n)} = \eta_P^{(n)}\big|_P\}_{n\in\mathbb{N}}$ of $F_P^{(-)}$ with respect to the pairing given by c_P (see § 4.1). Then we see that $\mathscr{F}_X^\vee$ is the completion of the space $k[[u_P^n]]$ of formal Taylor series in infinitely many variables $u_P^{(n)}$, $P \in X$, $n \in \mathbb{N}$. This realization of $\mathscr{F}_X^\vee$ is used to prove the following main result in the QFT of additive bosons.

Theorem 6. *There is a unique linear functional* $\langle\cdot\rangle\colon \mathscr{F}_X \to k$ *(the expectation value functional) with the following properties*:

EV1) $\langle \mathbf{1}_X\rangle = 1$,

EV2) $\langle \mathbf{a}\cdot v\rangle = 0$ *for all* $a \in \mathcal{A}(X, D_{\mathrm{ns}})$ *and* $v \in \mathscr{F}_X$.

The functional $\langle\cdot\rangle$ *is given by*

$$\langle v\rangle = (\Omega_X, v),$$

where

$$\Omega_X = \exp\biggl\{-\frac{1}{2}\sum_{m,n=1}^{\infty}\sum_{P,Q\in X} c_{PQ}^{(mn)} u_P^{(m)} u_Q^{(n)}\biggr\} \in \mathscr{F}_X^\vee,$$

$$c_{PQ}^{(mn)} = -\operatorname{Res}_Q(\eta_P^{(m)} d\eta_Q^{(n)}).$$

Proof. It follows from the decomposition (10) that a linear functional of the form $\langle v\rangle = (\Omega, v)$ possesses properties EV1) and EV2) if and only if it is normalized, $(\Omega, \mathbf{1}_X) = 1$, and $\Omega \in \mathscr{F}_X^\vee$ satisfies the system of equations

$$\Omega\cdot\boldsymbol{\eta}_P^{(n)} = 0 \tag{18}$$

for all $P \in X$ and $n \in \mathbb{N}$, where $\boldsymbol{\eta}_P^{(n)} = \rho^\vee(\eta_P^{(n)})$. Write $\eta_P^{(n)} = \beta_P^{(n)} + \gamma_P^{(n)}$, where $\beta_P^{(n)} = \{\beta_{PQ}^{(n)}\}_{Q\in X}$ and $\gamma_P^{(n)} = \{\gamma_{PQ}^{(n)}\}_{Q\in X} \in \mathbb{A}_X$ are given by

$$\beta_{PQ}^{(n)} = \begin{cases} 0 & \text{if } Q = P,\\ \eta_P^{(n)}\big|_Q & \text{if } Q \neq P,\end{cases} \qquad \gamma_{PQ}^{(n)} = \begin{cases} \eta_P^{(n)}\big|_P & \text{if } Q = P,\\ 0 & \text{if } Q \neq P.\end{cases}$$

It follows from (16) that $\boldsymbol{\gamma}_{\boldsymbol{P}}^{(\boldsymbol{n})}$ acts on $\mathscr{F}_X^\vee$ as differentiation with respect to $u_P^{(n)}$. For $Q \neq P$ we have

$$\beta_{PQ}^{(n)} = a_{PQ}^{(n)} + \sum_{m=1}^{\infty} a_{PQ}^{(nm)} u_Q^{(m)},$$

where $a_{PQ}^{(n)} \in k$ and

$$a_{PQ}^{(nm)} = c(\beta_{PQ}^{(n)}, v_Q^{(m)}) = -\operatorname{Res}_Q(\eta_P^{(n)} d\eta_Q^{(m)}) = c_{PQ}^{(nm)}.$$

Since $c_{PP}^{(nm)} = 0$ (see Lemma 1), we conclude that $\boldsymbol{\beta}_{\boldsymbol{P}}^{(\boldsymbol{n})}$ acts on $\mathscr{F}_X^\vee$ as multiplication by $\sum_{Q\in X} c_{PQ}^{(nm)} u_Q^{(m)}$. One can rewrite the equations (18) in the form

$$\biggl(\frac{\partial}{\partial u_P^{(n)}} + \sum_{Q\in X} c_{PQ}^{(nm)} u_Q^{(m)}\biggr)\Omega = 0, \qquad P \in X, \quad n \in \mathbb{N}. \tag{19}$$

It follows from part (i) of Lemma 1 that

$$c_{PQ}^{(mn)} = c_{QP}^{(nm)},$$

whence the system of differential equations (19) is compatible and Ω_X is its unique normalized solution. □

Remark 20. Let $\mathfrak{g}$ be a semisimple Lie algebra over k with the Cartan–Killing form $\langle\,,\,\rangle$. Then the k-vector space $\mathbb{A}_X$ with bilinear form c_X may be replaced by the k-vector space $\mathbb{V}_X = \mathfrak{g} \otimes_k \mathbb{A}_X$ with bilinear form $-\sum_{P\in X} \operatorname{Res}_P \langle x_P, dy_P\rangle$. Theorem 6 extends to this case. The additive Ward identities hold for $\mathfrak{g} \otimes_k \mathcal{A}(X, D_{\mathrm{ns}})$ and the corresponding QFT is associated with the current algebra on X in the sense of [6].

5.2. Charged additive bosons on X. The QFT of charged additive bosons is determined by the following data.

CB1) An effective non-special divisor $D_{\mathrm{ns}} = P_1 + \cdots + P_g$ of degree g on X with distinct points, uniformizers t_i at P_i and the k-vector space $\mathcal{A}(X, D_{\mathrm{ns}})$ of additive functions (a subspace of $\mathbb{A}_X$ containing $F = k(X)$) introduced in Example 1.

CB2) The local QFT's of charged additive bosons (the highest-weight $\mathfrak{l}_P$-modules $\mathscr{B}_P$ for all points $P \in X$).

CB3) The global lattice algebra $\mathfrak{l}_X$ (a semidirect sum of the global Heisenberg algebra $\mathfrak{g}_X$ and the Abelian Lie algebra $k[\mathrm{Div}_0(X)]$ with generators e_D, $D \in \mathrm{Div}_0(X)$, where $k[\mathrm{Div}_0(X)]$ is the group algebra of the additive group $\mathrm{Div}_0(X)$ of divisors of degree 0 on X).

CB4) A highest-weight $\mathfrak{l}_X$-module (the global Fock space $\mathscr{B}_X$ with highest-weight vector $\mathbf{1}_X \in \mathscr{B}_X$).

CB5) An expectation value functional, that is, a linear map $\langle\,\cdot\,\rangle\colon \mathscr{B}_X \to k$ with the following properties:

(i) $\langle \mathbf{e}_D \cdot \mathbf{1}_X\rangle = 1$ for all $D \in \mathrm{Div}_0(X)$,

(ii) $\langle \mathbf{a} \cdot u\rangle = 0$ for all $a \in \mathcal{A}(X, D_{\mathrm{ns}})$ and $u \in \mathscr{B}_X$.

As a k-vector space, the group algebra $k[\mathrm{Div}_0(X)]$ has a basis $\{e_D\}_{D\in \mathrm{Div}_0(X)}$, $e_{D_1} e_{D_2} = e_{D_1+D_2}$. For every $x = \{x_P\} \in \mathbb{A}_X$ and $D = \sum_{P\in X} n_P$, $P \in \mathrm{Div}_0(X)$, we put

$$x(D) = \sum_{P\in X} n_P x_P(0) \in k,$$

where $x_P(0) = x_P^+ \mod \mathfrak{p} \in k$ is the constant term of $x_P \in F_P$ (it is determined by the decomposition (12) associated with the non-special divisor D_{ns}; see § 4.2).

Definition 8. The *global lattice algebra* $\mathfrak{l}_X$ is the semidirect sum of the global Heisenberg algebra $\mathfrak{g}_X$ and the Abelian Lie algebra $k[\mathrm{Div}_0(X)]$ with Lie bracket

$$[x + \alpha C + \gamma e_{D_1}, y + \beta C + \delta e_{D_2}] = c_X(x, y)C + y(D_1)\gamma e_{D_1} - x(D_2)\delta e_{D_2},$$

where $x + \alpha C, y + \beta C \in \mathfrak{g}_X$ and $\gamma, \delta \in k$.

The global Fock space $\mathscr{B}_X$ is the tensor product of the group algebra $k[\mathrm{Div}_0(X)]$ and the Fock space $\mathscr{F}_X$ of additive bosons:

$$\mathscr{B}_X = k[\mathrm{Div}_0(X)] \otimes \mathscr{F}_X = \bigoplus_{D\in \mathrm{Div}_0(X)} \mathscr{B}_X^D,$$

where $\mathscr{B}_X^D = k \cdot e_D \otimes \mathscr{F}_X$. $\mathscr{B}_X$ is an irreducible $\mathfrak{l}_X$-module where $k[\mathrm{Div}_0(X)]$ acts by multiplication:

$$\mathbf{e}_{D_1}(e_{D_2} \otimes v) = e_{D_1+D_2} \otimes v, \qquad v \in \mathscr{F}_X, \tag{20}$$

and $\mathbb{A}_X$ acts by the formula

$$\mathbf{x}(e_D \otimes v) = -x(D)e_D \otimes v + e_D \otimes (\mathbf{x} \cdot v), \qquad x \in \mathbb{A}_X, \quad v \in \mathscr{F}_X. \tag{21}$$

For every $D = \sum_{P \in X} n_P P \in \mathrm{Div}_0(X)$, the subspace $\mathscr{B}_X^D$ is an irreducible $\mathfrak{g}_X$-module with the following property. If $x = \{x_P\}_{P \in X} \in \mathbb{A}_X$ with $x_P \in k$ for all $P \in X$, then the restriction of the operator $\mathbf{x}$ to $\mathscr{B}_X^D$ is equal to $-x(D)\mathbf{I}$, where $\mathbf{I}$ is the identity operator. In particular, when $x = c$ is a constant, we have $x(D) = c \deg D = 0$ and $\mathbf{x}$ acts by zero on $\mathscr{B}_X$.

Remark 21. One can also define an extended lattice algebra $\widetilde{\mathfrak{l}}_X$ as a semidirect sum of the global Heisenberg algebra $\mathfrak{g}_X$ and the Abelian Lie algebra $k[\mathrm{Div}(X)]$. The corresponding irreducible $\widetilde{\mathfrak{l}}_X$-module is the extended Fock space

$$\widetilde{\mathscr{B}}_X = k[\mathrm{Div}(X)] \otimes \mathscr{F}_X = \bigoplus_{D \in \mathrm{Div}(X)} \mathscr{B}_X^D.$$

The action of $\widetilde{\mathfrak{l}}_X$ on $\widetilde{\mathscr{B}}_X$ is given by the same formulae (20), (21), where now the constant adèle $x = c$ acts on $\mathscr{B}_X^D$ as $(c \deg D)\mathbf{I}$.

The dual Fock space $\mathscr{B}_X^\vee$ is defined as a completion of the direct sum of the dual spaces to $\mathscr{B}_X^D$ over $D \in \mathrm{Div}_0(X)$. This completion is given by formal infinite series. Explicitly,

$$\mathscr{B}_X^\vee = \overline{\bigoplus_{D \in \mathrm{Div}_0(X)} \mathscr{B}_X^\vee(D)},$$

where

$$\mathscr{B}_X^\vee(D) = k \cdot q^D \otimes \mathscr{F}_X^\vee,$$

$q^D \in k[\mathrm{Div}_0(X)]^\vee$ is dual to e_D, and $\mathscr{F}_X^\vee$ was defined in § 5.1.

Theorem 7. *There is a unique linear functional* $\langle \cdot \rangle \colon \mathscr{B}_X \to k$ *(the expectation value functional) with the following properties*:

EV1) $\langle \mathbf{e}_D \cdot \mathbf{1}_X \rangle = 1$ *for all* $D \in \mathrm{Div}_0(X)$,

EV2) $\langle \mathbf{a} \cdot v \rangle = 0$ *for all* $a \in \mathcal{A}(X, D_{\mathrm{ns}})$ *and* $v \in \mathscr{B}_X$.

The functional $\langle \cdot \rangle$ *is given by*

$$\langle v \rangle = (\widehat{\Omega}_X, v),$$

where

$$\widehat{\Omega}_X = \sum_{D \in \mathrm{Div}_0(X)} q^D \otimes \exp\left\{ \sum_{n=1}^{\infty} \sum_{P \in X} \eta_P^{(n)}(D) u_P^{(n)} \right\} \Omega_X \in \mathscr{B}_X^\vee$$

and Ω_X *is defined in Theorem* 6.

Proof. We put

$$\Omega = \sum_{D\in \mathrm{Div}_0(X)} q^D \otimes \Omega_D, \qquad \Omega_D \in \mathscr{F}_X^\vee.$$

The condition $(\Omega, e_D \otimes \mathbf{1}_X) = 1$ for all $D \in \mathrm{Div}_0(X)$ is equivalent to the normalization $(\Omega_D, \mathbf{1}_X) = 1$. Since the constants act by 0 on $\mathscr{B}_X$, it suffices to verify that

$$(q^D \otimes \Omega_D) \cdot \boldsymbol{\eta}_{\boldsymbol{P}}^{(\boldsymbol{n})} = 0 \tag{22}$$

for all $D = \sum_{Q\in X} n_Q\, Q \in \mathrm{Div}_0(X)$ and $P \in X$. Since

$$q^D \cdot \boldsymbol{\eta}_{\boldsymbol{P}}^{(\boldsymbol{n})} = -\eta_P^{(n)}(D)\, q^D = -\sum_{Q\in X} n_Q \eta_P^{(n)}\Big|_Q (0)\, q^D$$

(note that $\eta_P^{(n)}\big|_P(0) = 0$ by the definition in § 4.2), we see from (22) that Ω_D satisfies the following system of differential equations:

$$\left(\frac{\partial}{\partial u_P^{(n)}} - \sum_{Q\in X} n_Q \eta_P^{(n)}\Big|_Q (0) + \sum_{Q\in X} c_{PQ}^{(nm)} u_Q^{(m)}\right)\Omega_D = 0.$$

This system has the unique normalized solution

$$\Omega_D = \exp\left\{\sum_{n=1}^{\infty}\sum_{P\in X} \eta_P^{(n)}(D) u_P^{(n)} - \frac{1}{2}\sum_{m,n=1}^{\infty}\sum_{P,Q\in X} c_{PQ}^{(mn)} u_P^{(m)} u_Q^{(n)}\right\}. \qquad \square$$

Remark 22. Theorems 6, 7 hold for an arbitrary field k of constants of characteristic 0 (see Remark 10).

Remark 23. All results in this section hold trivially in the case when X has genus 0. Using Remark 8, one can easily obtain elementary explicit formulae for the expectation value functional $\langle\cdot\rangle$ for quantum additive and charged bosons on $\mathbb{P}^1_k$.

5.3. Invariant formulation. Here we present an invariant formulation and a proof of a generalization of Theorem 6 for the current algebra. They were suggested by the referee. Let V be a k-vector space regarded as an Abelian Lie algebra, and let c be a skew-symmetric bilinear form on V. We write $\widetilde{V}$ for the one-dimensional central extension of V

$$0 \to k\cdot C \to \widetilde{V} \to V \to 0$$

by the 2-cocycle c, and $\mathscr{W}$ for the Weyl algebra of the Lie algebra $\widetilde{V}$, as in § 4.1. Let U and W be isotropic subspaces of V with respect to c such that $U \cap W$ and $V/(U+W)$ are finite-dimensional and $U \cap W$ lies in the kernel of c.

Lemma 2. *There is a canonical isomorphism of k-vector spaces*

$$\mathscr{W}/\mathscr{W}\cdot(U+W) \simeq \mathrm{Sym}^\bullet\big(V/(U+W)\big).$$

Proof. This is proved by direct calculation in a symplectic basis of V compatible with the corresponding bases in U and W. $\square$

In the notation of Remark 20 we put $V = \mathbb{V}_X = \mathfrak{g} \otimes_k \mathbb{A}_X$,

$$c(x, y) = -\sum_{P \in X} \mathrm{Res}_P \langle x_P, dy_P \rangle,$$

$U = \mathfrak{g} \otimes_k F$ and $W = \mathfrak{g} \otimes_k \mathbb{O}_X$, where $F = k(X)$. Then

$$\mathscr{W}/\mathscr{W} \cdot W \simeq \mathscr{F}_X$$

is the Fock space of the current algebra on X. Using Serre's adelic interpretation of cohomology in the form

$$V/(U + W) \simeq \mathfrak{g} \otimes_k H^1(X, \mathcal{O}_X),$$

we obtain from Lemma 2 that

$$\mathscr{F}_X/(\mathfrak{g} \otimes_k F) \cdot \mathscr{F}_X \simeq \mathrm{Sym}^\bullet(\mathfrak{g} \otimes_k H^1(X, \mathcal{O}_X)).$$

This shows that the global symmetries $\mathfrak{g} \otimes_k F$ do not uniquely determine the expectation value functional $\langle \cdot \rangle$ except in the case when $X = \mathbb{P}^1_k$.

To extend the Lie algebra of global symmetries, we consider a Lagrangian subspace $L \subset H^1_{\mathrm{dR}}(X) \simeq \Omega^{(2\mathrm{nd})}/dF$ such that the restriction to L of the natural map $H^1_{\mathrm{dR}}(X) \to H^1(X, \mathcal{O}_X)$ is an isomorphism. For example, take $L = k \cdot \omega_1 \oplus \cdots \oplus k \cdot \omega_g$ (see Theorem 4). Let $\widetilde{L}$ be the inverse image of L under the map $\Omega^{(2\mathrm{nd})} \to \Omega^{(2\mathrm{nd})}/dF$. We claim that there is a subspace $U_0 \subset \mathbb{A}_X$ such that

$$F \subset U_0, \qquad U_0 \cap \mathbb{O}_X = k, \qquad dU_0 = \widetilde{L}.$$

For example, take $U_0 = \mathcal{A}(X, D_{\mathrm{ns}})$. Then $U = \mathfrak{g} \otimes_k U_0$ is an isotropic subspace of V and, by Remark 9, we have

$$V/(U + W) = \{0\}.$$

Therefore,

$$\mathscr{F}_X/U \cdot \mathscr{F}_X \simeq k,$$

which is essentially Theorem 6 (without an explicit formula for the vector Ω_X). By Remark 10, the condition that the field k is algebraically closed is not necessary.

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Received 11/OCT/11
19/APR/12
Translated by THE AUTHOR